Assignment 3

Hand in no. 1, 4, 7 and 8 by October 3, 2023.

1. Let f be a 2π-periodic function integrable on $[-\pi, \pi]$ whose Fourier series is the zero function. Show that

   (a)
   $$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0 \ ,$$
   for all continuous, 2π-periodic functions g.

   (b)
   $$\int_{-\pi}^{\pi} f(x)s(x)\,dx = 0 \ ,$$
   for all step functions s, and

   (c) Deduce that $f = 0$ almost everywhere.

2. Show that the "Fourier map" $f \mapsto \hat{f}(n) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ satisfies $\hat{f} = \hat{g}$ if and only if $f = g$ almost everywhere.

3. Prove Hölder's Inequality in vector form: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $p > 1$ and q conjugate to p,
   $$|\mathbf{x} \cdot \mathbf{y}| \le \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |y_j|^q\right)^{1/q} .$$
   You may prove it directly or deduce it from its integral form by choosing suitable functions f anf g.

4. Prove Minkowski's Inequality in vector form: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $p > 1$,
   $$\|\mathbf{x} + \mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p \ .$$
   You may prove it directly or deduce it from its integral form by choosing suitable functions f anf g.

5. Prove the generalized Hölder's Inequality: For $f_1, f_2, \cdots, f_n \in R[a, b]$,
   $$\int_a^b |f_1 f_2 \cdots f_n| dx \le \left(\int_a^b |f_1|^{p_1}\right)^{1/p_1} \left(\int_a^b |f_2|^{p_2}\right)^{1/p_2} \cdots \left(\int_a^b |f_n|^{p_n}\right)^{1/p_n} ,$$
   where
   $$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1, \quad p_1, p_2, \cdots, p_n > 1 \ .$$

6. Show that for $\mathbf{x} \in \mathbb{R}^n, 1 \le p < r \le \infty$,

   (a)
   $$\|\mathbf{x}\|_p \le n^{\frac{1}{p} - \frac{1}{r}} \|\mathbf{x}\|_r \ ,$$

   (b)
   $$\|\mathbf{x}\|_r \le n^{\frac{1}{r}} \|\mathbf{x}\|_p .$$

7. Show that for $1 \leq p < r \leq \infty$, and $f \in C[a,b]$,

$$\|f\|_p \leq (b-a)^{\frac{1}{p}-\frac{1}{r}} \|f\|_r \ .$$

8. Show that there is no constant C such that $\|f\|_2 \leq C\|f\|_1$, for all $f \in C[0,1]$.

9. Show that $\|\cdot\|_p$ is no longer a norm on $C[0,1]$ for $p \in (0,1)$.

10. In a metric space (X,d), its metric ball is the set $\{y \in X : \ d(y,x) < r\}$ where x is the center and r the radius of the ball. May denote it by $B_r(x)$. Draw the unit metric balls centered at the origin with respect to the metrics d_2, d_∞ and d_1 on $\mathbb{R}^2$.